Abstract

This book is aimed at pre-university students and its purpose is to contribute to the development of their knowledge related to the algebraic and transcendent equations studied at school, as well as their application to different situations that occur in practice in an innovative and creative way, using the procedures for solving them, so that it allows the consolidation of attitudes such as industriousness, responsibility and science.

The system of knowledge worked on and treated didactically in this book is related to the algebraic equations and within them the linear, quadratic, fractional and radical equations, the modular equations and the transcendental equations such as, the exponential, logarithmic and trigonometric equations, providing the minimum theoretical and methodological resources, necessary to learn and to successfully face the exercises and problems proposed in each chapter.

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Methodology of Mathematics Teaching
Treatment to School Mathematics Equations

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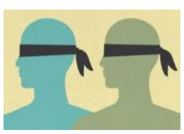

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I- Brief historical review of the equations

Since ancient times, and as an essential part of his own evolutionary development, man has tried to understand the different aspects that are part of his daily life. To do so, he has tried to have tools that allow him not only to hunt and gather more efficiently, but also to measure lengths, order and count objects, or recognize periodic phenomena in nature. As part of this elaboration process, man has built models that have facilitated the task of solving specific problems or that have helped him find a solution to the specific problem that affects him. All this with the purpose of favoring both his way of life and that of the members of his community. Many of these situations can be posed by certain equations with coefficients in some numerical domain and with a few variables or unknowns. Remember that the word *equation* comes from the Latin *aequatio* which means *equality*.

The first rudiments about the equations have been found in the oldest mathematical document that has reached our days: the Rhind papyrus, preserved in the British Museum with some fragments in the Brooklyn Museum, and also known as the Book of Calculus, which was written by the Egyptian priest Ahmés around 1650 BC and exhumed in Thebes in 1855.

On the one hand, Chinese mathematicians during the third and fourth centuries B.C. continued the tradition of the Babylonians and bequeathed to us the first methods of linear thought. For example, in the treatise Nine Chapters on Mathematical Art, published during the Han Dynasty, the following linear system appears:

$$\begin{aligned} 3x+2y+z&=39\\ 2x+3y+z&=34\\ x+2y+3z&=26 \end{aligned}$$

As well as a method for its resolution, known as the "fan-chen" rule, which, in essence, is the well-known Gauss method of our days.

In this book, the study of the equations is initiated, which is one of the fundamental topics of the Linear Algebra, so much for its application in the development of other topics of the same one, as in other branches of the science.

The importance of the study of the equations is immediately understood, as soon as it is analyzed that a great part of problems is modeled by means of equations and the solution of these problems is indissolubly conditioned to the resolution of these mathematical objects.

II. Preliminary considerations

Equations are calculations that contain variables and only become true propositions for certain values of the variable. The goal is to discover those values.

Examples of equations

a) 3x + 4 = x + 10 b) $2x^2 - 2x = 7$ c) $\frac{2x-1}{x+3} = 5$ d) $2^{x+4} = 8$

Later on, you will deepen in the procedure for the solution of these equations and others that will be studied.

We call propositional form to a linguistic structure that contains at least one variable and that becomes a proposition when all the variables are substituted by symbols, which denote objects of the domain, so we can also call equation to propositional form $T_1 = T_2$, where T_1 and T_2 are mathematical expressions constituted by terms.

For example, 3x+4=x-2 is a propositional form

It is necessary to emphasize that a term is a number, or a variable or the combination of numbers and variables using the operations of multiplication, division and empowerment.

Let's see then, some examples of terms

a-) 2 , b-) x^2 , c-) $4x^3y$, d-) $\frac{3m^2z}{xy}$

The following examples do not constitute terms

a-)$2x - y$, b-) $\frac{4x^2y^2}{x-y}$

Since the equations are equal, everything on the left of the equation sign is called the left member and everything on the right is called the right member.

So, in the equation 5x-8=2x+3 the left member is 5x-8 and the right member is 2x+3

In the equations, the variables are replaced by values of the numerical sets. These sets of numbers are called the **domain of the variable**. You already know that the statements that can be assigned a truth value, that is, that they can only be true or false, are propositions. If by substituting the variable for a value of its domain, the equation is transformed into a true proposition, then it is said that that number by which the variable was substituted is a **solution of the equation**.

The set formed by all the solutions of the equation is called a **solution set**.

Hence to solve an equation, it is necessary to find the value of the domain of the variable that transforms the equation into a true proposition, the solution of the equation depends on the domain of the variable; that's why, there are equations that can have solution in a numerical set and not have solution in another one.

Equations with equal domain of the variable that have the same set solution are called **equivalent equations**. Then the equations 2x = 6, x+3 = 6, for example, have the same domain of the variable and the same solution set so they are **equivalent equations**.

The transformations made to an equation, which as a result lead to obtaining equations equivalent to the given one, are called equivalent transformations.

Equivalent transformations:

1-Switching the members of the equation

For example, the equations -7 + 4x = 6x + 2 and 6x + 2 = - 7 + 4x are equivalent, as they are obtained from each other by the exchange of their members.

2-Adding (subtracting) the same term to both members of an equation

For example, in the equation 6x + 2 = - 7 + 4x if we add - 2 to both members of the equation we get 6x + 2 - 2 = - 7+ 4x - 2, that is, the equation 6x = - 9 + 4x, which has the same set solution as the initial equation.

3-Multiplying (dividing) both members of the equation by a number other than zero.

For example, in the equation 2x = - 9 if we divide both members by 2 we get the equation $x = -\frac{9}{2}$, which is equivalent to the previous one. Notice that dividing by 2 both members of the equation has cleared the variable x.

4-Suppressing grouping signs.

5-Placing grouping signs.

6-Simplifying and expanding numerical fractions.

7-Applying the commutative law of addition and multiplication in different numerical domains.

8 - Grouping terms (addition, multiplication, etc.).

Equivalent transformations are used to solve the equations, since their application allows us to transform the different equations to an equivalent of the form x = b, thus determining the solution of the equation.

It is important to emphasize that every equation that is transformed into a false proposition for every value assigned to the variable has no solution or simply its solution is empty, and if it is transformed into a true proposition for every value assigned to the the variable it has infinite solutions.

In a Mathematics class the students are proposed to solve the following situation:

The speed of a river's current is 3.0 km/h. A boat takes the same time to navigate 8.0 km downstream as it does to navigate 5.0 km upstream. What is the speed of the boat in calm water?

Based on the conditions of the situation posed, let us analyze the following questions:

1. How can the identified unknown be designated through mathematical language?

2. Can the speed of the mobile be expressed by means of an expression containing only one variable?

3. What relationships or combinations are involved in these?

4. Which is the subset of real numbers that contains exactly all the values that allow, to the expressions that represent the times, to take a value determined univocally?

5. What mathematical model allows to solve the declared situation?

How is the given model solved?

Does the solution obtained satisfy the requirements of the problem?

The answers to the previous questions will be constructed as far as the theoretical aspects of the book would be advanced.

Chapter 1. Algebraic Equations

1.1 The linear equation

Definition: An equation is called a **linear equation in one variable** if and only if it can be reduced by the transformations equivalent to the form $ax+b=0$ with a, b real numbers and $a \neq 0$.

Examples of linear equation

a-3x-2=0 b- 4x=0 c- 3x+6=x-8

Is the equation 3(2x-1) =6x-3 a linear equation?

Obviously not, because through equivalent transformations it is not possible to obtain an equation of the form with $ax+b=0$ a, b real numbers and $a \neq 0$.

When carrying out the transformations a true proposition 0 = 0 is obtained, so the equality is fulfilled for the whole set of values belonging to the domain of definition of the equation.

1.1.1. Procedure for solving a linear equation

1 **-** Eliminate signs of grouping and reduce similar terms in each of the members of the equation.

2-Transpose all the terms to a member of the equation obtaining zero in one of the members of the equation.

3-Reduce similar terms until obtaining the equation of the form $ax+b=0$ and $a \neq 0$.

4-Clear the variable

5- Solution Set

Note: In linear equations it is not necessary to perform the verification because in its solution process no strange solutions are introduced, it is only performed if it is suggested to do it or if you need to verify that you have not made any mistake during the solution process.

Examples

Solve the following equations

a) $5x+7=3(x+1)$

$5x+7=3x+3$ Eliminating parentheses

$5x+7-3x-3=0$ Transposing all the terms to the left member of the equation.

$2x + 4 = 0$	Reducing similar terms and obtaining the form of the linear equation.
$x = \frac{-4}{2}$	Clearing the variable.
$x = -2$	Operating and finding the solution.

The equation has as a set solution $S = \{-2\}$.

b) $4(x+1) - x = 2x + 3 - x$

$4x + 4 - x = 2x + 3 - x$	Eliminating parentheses.
$3x + 4 = x + 3$	Reducing similar terms.
$3x + 4 - x - 3 = 0$	Transposing all the terms to the left member of the equation.
$2x + 1 = 0$	Reducing similar terms and obtaining the form of the linear equation.
$x = -\frac{1}{2}$	Clearing the variable.

The set solution of the equation is $S = \left\{-\frac{1}{2}\right\}$

Note. If you are solving an equation that by simple inspection **seems** to be a linear equation you must verify that it is, it is not enough that the highest exponent of the variable is 1 because for example the equation 2x-3=2x+5, seems to be linear, but in fact it is not.

There are teachers and students who when solving a linear equation follow the following path (previous example)

$4(x+1) - x = 2x + 3 - x$

$4x + 4 - x = 2x + 3 - x$	Eliminating parentheses.
$3x + 4 = x + 3$	Reducing similar terms.
$3x - x = 3 - 4$ (*)	**Transposing the variables to one member and the independent terms to the other member of**

	the equation.
$2x = -1$	Reducing similar terms.
$x = -\frac{1}{2}$ (**)	Clearing the variable.

Although the result is the same, when performing step (*), knowledge is not correctly formalized, a necessary element in the development of mathematical logical thinking, so it is suggested that you do it only when you have the concept of linear equation well rooted.

Let's see an example that illustrates the above mention statement

Solve the following equation

$$8x + 2(x - 1) = 3x(x + 1)$$

The equation proposed by simple inspection seems to be linear then we must apply the procedure to solve an equation of this type

$8x + 2(x - 1) = 3x(x + 1)$	
	Eliminating parentheses.
$8x + 2x - 2 = 3x^2 + 3x$	
	Reducing similar terms in the left member.
$10x - 2 = 3x^2 + 3x$	
$-2 = 3x^2 + 3x - 10x$	Transposing the variables to one member and the independent terms to the other as was done in (*)
$-2 = 3x^2 - 7x$	Reducing similar terms (observe that here it is not so simple to clear the variable as in (**) of the previous example.

Of course, if you try to take this resulting equation to the structure of a linear equation, as required by the definition, you will notice that it is not possible because you get $3x^2 - 7x + 2 = 0$ which obviously does not have the structure of a linear equation

Then, the equation $8x + 2(x-1) = 3x(x+1)$ is not a linear equation.

Let's see another example

$2a – (a +2) (a + 5) = 6 – a (a – 3)$

This equation seems to be linear so the procedure to be followed to solve a linear equation should be applied initially

$2a – (a^2 +7a + 10) = 6 – a^2 + 3^a$	Carrying out the indicated operations
$2a – a^2 – 7a – 10 = 6 – a^2 + 3a$	Eliminating parentheses (**this equation now seems to be quadratic, so from now on the procedure to be followed is the one for solving a quadratic equation, which we will see it later, but we advance that all the terms must be transposed to a member and equal them to zero**)
$2a - a^2 - 7a - 10 - 6 + a^2 - 3a = 0$	
$-8a - 16 = 0$	Reducing similar terms in the left member (note that what seemed initially is not, since by performing equivalent transformations we arrived at the structure of a linear equation)
$– 8a = 16$	Clearing the variable.
$a = – 2$	

Then the set solution is $S = \{-2\}$.

1.1.2 Exercises Proposal

1- Solve the following equations

a-) $3+\sqrt{\frac{4}{5}}=x+5$ b-) $3(x-2)-4x=3(x-1)$ c-) $3x-4=2x-1$

d-) $6x+215$ e-) $6a=24$ f-) $100=10z$ g-) $3x=7$ h-) $6z=0$

i-) $5(x-3)=3(x+1)$ j-) $100=-10z$ k-) $11x-88=0$

l-) $5x-29=2x-7$ m-) $x-6=7x$ n-) $x^2+3x-4=(x+1)^2$

ñ-) $3x+2=12-2x$ o-) $3x+3=12-2x$ p-) $5a=9a-24$

q-) $6(2x-5)+2=20$ r-) $5(x+3)+2(x-9)=4$ s-) $5(7x-8)=3(x+8)-4(6x-7)$

t-) $(12x+7)(12x-7)=(12x-9)^2+86$ u-) $(9x-3)^2+(12x-4)^2=(15x-10)^2$

v-) $[3+(x+2)]^2=x^2+5$ w-) $(x-5)^3+1=(x+4)(x-4)(x-15)$

y-) $(x-1)^3=(x+2)^3-9(x^2-1)$ z-) $\frac{2x+1}{3}-\frac{x-1}{5}=\frac{7x-2}{15}$ ρ-) $\frac{x+3}{2}-\frac{x-1}{3}=\frac{x+5}{6}+1$

π-) $3{,}2m-(4m+3)=2{,}4(3m-1)$

2- Determine the value of x in the following equations:

d-) $x+ax=b$ e-) $ax+bx=c$ f-) $ax+bx=cx+1$ g-) $a(x-a)=4(x-4)$

h-) $m^2x-3=m+9x$ i-) $b^2x-2=b+4x$ j-) $a(x+b)=a^2+b^2+b(x-a)$

3- Determine an equation that is equivalent to the given equations.

a) $5(x-3)=3(x+1)$; $x\in Q$ b) $100=-10z$; $z\in Z$

4-In which basic domain does the equation not have a solution $-3x=9$?

5-Find a number such that 6 plus half the number is two-thirds of the number.

6-The sum of three consecutive natural numbers is equal to four times the smallest. What are the numbers?

7-A number is such that its triple minus 8 units is 37, what number is it?

8-Find four consecutive even integers so that the sum of the first three numbers is twice as big as the double of the fourth number.

9- I need a dollar for buying my favorite computer magazine. If I had twice what I have now, I would have 2.10. How much do I have? How much does the magazine cost?

10- 30*Kg* of coffee $3.60\$/Kg$ are mixed with a certain amount of a better coffee $4.8\$Kg$ having as a result a mixed of $4.35\$/Kg$. What is the specific amount of better coffee being used?

11- In a rectangle with a perimeter of 54m, the length of the length is less than twice its width. How long are the sides of the rectangle?

12- In a triangle one of its sides has a length of 11cm, and the other two sides are of lengths that represent a fifth and a quarter of the length of the perimeter of the triangle. What is the length of the perimeter of the triangle?

13-The price of a bicycle after discounting 20% is $72CUC. How much did the bicycle cost before the discount?

14-A self-employed person sells an item that is 60% more expensive than it costs to wholesale. How much does an item with a wholesale price of $144 cost?

15-A pig farm worker is paid a base salary of $2150 per month, plus an 8% commission if he sells more than $7000 during that period. How much must he sell to earn $3170 per month?

16-How many liters of a mixture containing 80% alcohol would have to be added to 5 liters of a 20% solution to obtain a 30% solution?

17- How many gallons of distilled water would have to be mixed with 50 gallons of a 30% solution with 30% alcohol to obtain a 25% solution?

18- How many gallons of hydrochloric acid must be added to 12 gallons of a 30% solution to obtain a 40% solution?

1.2 Quadratic or second-degree equations

Definition. An equation is called a second-degree or quadratic equation with exactly one variable, if and only if it can be brought, by equivalent transformations, to the form:

$$ax^2 + bx + c = 0 \qquad (a \in R, b \in R, c \in R \wedge a \neq 0)$$

where a y b are the coefficients of the quadratic and linear terms respectively and c the independent term.

It should be noted that in the quadratic equation if $b \neq 0$ y $c \neq 0$, the equation is said to be complete, and if $b = 0$ o $c = 0$ the equation is incomplete.

Example of quadratic equations in the basic solution domain R

a) $x^2 - 4x = 0$
b) $x^2 + 2x - 1680 = 0$
c) $4x^2 - 9 = 0$
d) $x^2 + 3 = 0$

To obtain the set solution of a quadratic equation with exactly one variable, the following cases can be considered: factorization, square root, the general formula of the discriminant and completing the square.

1.2.1 Procedures for solving quadratic equations

Case 1: Factorization

In the above example equation a) is incomplete. It can be solved by applying the common factor in the left term of the equation:

$$x^2 - 4x = 0$$
$$x(x - 4) = 0$$

But the product of two numbers is 0 if and only if at least one of the factors is 0, from where it has to be:

$$x = 0 \text{ o } x - 4 = 0$$

And as a set solution

$$S = \{0;4\}$$

Equation b) is complete. A factorial decomposition is tested for a and c(a=m.n, c=p.q) by arranging the factors in two columns:

a	*c*
m	*p*
n	*q*

If $mq + np = b$ (Vieta's theorem), then the factors are $mx + p$ y $nx + q$.

Then in the example you have to

$$x^2 + 2x - 1680 = 0$$

$$(x+42)(x-40) = 0$$

$$x = -42 \quad \text{o} \quad x = 40$$

And as a set solution $S = \{-42, 40\}$

For equation c), the left term is a difference of squares, therefore of:

$$4x^2 - 9 = 0$$

$$(2x+3)(2x-3) = 0 ,$$

$$x = -\frac{3}{2} \quad \text{o} \quad x = \frac{3}{2}$$

Case 2: Square root method

This method requires the use of the property mentioned below.

For any real number k , $k > 0$, the equation $x^2 = k$ is equivalent to :

$$x = \pm\sqrt{k}$$

if $k = 0$ then $x = 0$

Equation c), from the previous example, can also be solved as follows:

$$4x^2-9=0$$

$4x^2=9$, then $x^2=\frac{9}{4}$. Hence

$x=\pm\sqrt{\frac{9}{4}}$, Therefore $S=\left\{-\frac{3}{2};\frac{3}{2}\right\}$.

In equation d) you have to

$x^2+3=0$ from here you have to

$x^2=-3$. as $x^2\geq 0$, the equation has no solution in R

$S=\varnothing$

Case 3: General formula of the discriminant

The equation $ax^2+bx+c=0$ $(a,b,c\in R\wedge a\neq 0)$ has real solutions yes and only yes $D=b^2-4ac\geq 0$. Si D > 0, then the equation has two solutions

$$x_1=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \quad y \quad x_2=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$$

if $D=0$, then the equation has a solution $x=-\frac{b}{2a}$.

The expression $D=b^2-4ac$ is called the discriminant of the second-degree equation.

if $D<0$, then the equation has no real solutions so it is concluded that the equation has an empty solution.

Example

Solve the following quadratic equations in the domain of real numbers using the general formula of the discriminant:

1) $x^2+2x-1680=0$

This equation coincides with point b already solved in case 1. In this case $a=1, b=2, c=-1680$. Let's analyze if the given equation has solutions in the given domain.

To do this, let's analyze the value of the discriminant.

$D=b^2-4ac=2^2-4\cdot 1\cdot(-1680)=4+6720=6724$, such as 40 is greater than zero the equation has two real solutions.

$$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-2\pm\sqrt{6724}}{2}=\frac{-2\pm 82}{2}$$

$x_1 = 40$ y $x_2 = -42$

2) $9x^2+6x+1=0$

In this case $a=9, b=6, c=1$. Let's analyze if the given equation has solutions in the given domain.

$D=b^2-4ac=6^2-4\cdot 9\cdot 1=0$

As the discriminant is equal to zero then the given equation has only one real solution, therefore $x=\frac{-6\pm 0}{2\cdot 9}=-\frac{6}{18}=-\frac{1}{3}= -\frac{6}{18}=-\frac{1}{3}$

$$S=\left\{-\frac{1}{3}\right\}$$

3) $5x^2-4x+1=0$

In this equation you have to $a=5, b=-4, c=1$. To determine the value of the discriminant you have to $D=(-4)^2-4\cdot 5\cdot 1=16-20=-4$. Such as $D<$ 0 the second-degree equation has no solution in the domain of real numbers.

Note: Any quadratic equation can be solved using the general formula of the discriminant.

Case 4: Completing the square

Let's analyze how one of the oldest artifices of mathematics, called the method of completing squares, works in this case.

Be the equation $ax^2+bx+c=0$ $(a,b,c\in R\wedge a\neq 0)$. We multiply each term in the equation by $4a$, obtaining

$$4a^2x^2+4abx+4ac=0$$

The value of the term independent in both members of the equality is subtracted:

$$4a^2x^2+4abx=-4ac$$

To complete the perfect square trinomial, or more briefly, to complete the square in the left member, we add the square of half the linear coefficient, so we add b^2 in both members of the equation:

$$4a^2x^2+4abx+b^2=b^2-4ac$$

We factor the perfect square trinomial on the left side and perform the indicated operation on the right side:

$$(2ax+b)^2=b^2-4ac$$

We extract square root in both members: $(2ax+b)=\pm\sqrt{b^2-4ac}$

We clear up the unknown that we are looking for: $2ax=-b\pm\sqrt{b^2-4ac}$ and we obtain the general formula $x_{1,2}=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$ of the second-degree equation.

Example

Solve the following equations by the method of completing the square:

1) $x^2+6x+7=0$

The square of the half of the linear coefficient is added to both members of the equation, that is, $\left(\frac{6}{2}\right)^2$, obtaining

$x^2+6x+\left(\frac{6}{2}\right)^2+7=\left(\frac{6}{2}\right)^2$. From here you have to

$x^2+6x+9=9-7$

$(x+3)^2=2$. Clearing the variable

$x_{1,2}=-3\pm\sqrt{2}$

1.2.2 Exercises Proposal

1-Solve the following equations

a-) $x^2=100$ b-) $x^2-225=0$ c-) $x^2-3c^2=0$ (variable x)

d-) $x^2=\frac{4}{9}m^2+mn+\frac{9}{16}n^2$ (variable x) e-) $x(2x-3)-3(5-x)=83$

f-) $2x^2+35=1315-3x^2$ (variable x) g-) $(2x+5)(2x-5)=11$

h-) $(7+x)^2+(7-x)^2=130$ i-) $8(2-x)^2=2(8-x)^2$

j-) $x^2+4ax-12a^2=0$ (variable x) k-) $x^2-5ax+6a^2=0$ (variable x)

l-) $abx^2+(a^2-b^2)x-2ab=0$ (variable x) m-) $a(x+a)^2=b(x+b)^2$ (variable x)

n-) $3x^2-(2x-3)=-(x+1)+2x^2$ ñ-) $2m^2+4(-m+1)=3(m^2+3)-4m^2$

o-) $x^2-x=-12$ p-) $x^2=-(5x+6)$ q-) $x^2=0{,}81$ r-) $\frac{1}{4}x^2=16$

2. What are the necessary conditions for m so that the equation: $x^2-\sqrt{m}x+4$ has only one solution?

3- Calculate $k\in R$ so that $\left(\frac{11-5k}{4}\right)x^2+(k-1)x+1=0,$ have your two solutions the same.

4- Be the equation $2x^2+(k+1)x+3(k-5)=0$. Find all the values that k must take in order for it to be true that all roots are real and equal.

5- The lengths of the legs of a right triangle differ in $7cm$. The area is $60m^2$. Calculate the perimeter of the triangle.

6- If the sides of a square are increased by $3m$ y $5m$, respectively, the surface of the new rectangle is $440m^2$. How long was the side of the primitive square?

7-The product of one integer by the next is 272. Calculate them

8. Find two consecutive even numbers whose product is 528.

7. By increasing the side of a square in $8cm$, your area increases by $128cm^2$. How tall was your side?

8. As it increases by $5m$ the side of a square, its area increases by $75m^2$. Calculate the side of the square.

9. In a right-angled triangle, the largest side is $3cm$ longer than the medium one, and this one, $3cm$ longer than the small one. How long are the sides?

10. In a right-angled triangle, a leg measures $2cm$ less than the hypotenuse and $14cm$ more than the other leg. It calculates the length of its three sides.

1.3 Fractional equations

Be T_1 y T_2 two terms in the same variable, a rational or fractional equation is an equality reducible to form:

$$\frac{T_1}{T_2} = 0$$

Examples :

a) $\frac{2x}{x+2} = 0$ b) $\frac{x^2-1}{5x-3} = 0$ c) $\frac{2}{3z-1} = 0$ d) $\frac{3x+7}{x-4} = 3$

Note: For the equation of the type $\frac{T_1}{T_2} = 0$ constitutes a fractional equation it is necessary that in the denominator T_2 is found the variable.

1.3.1 Procedures for solving a fractional equation

Procedure 1

1-Obtain the zero in one of the members of the equation as a result of transposing to the other member all the terms existing in it.

2-Obtain the equation of the form $\frac{T_1}{T_2} = 0$ performing the indicated operations

3- Solve the equation $T_1 = 0$ getting your solution

4-Check this solution obtained in the initial equation.

5-Write up the solution

Note: In the fractional equations it is mandatory to check

Let's see an example

Solve the following equation

$$\frac{3}{x-1} = \frac{6}{x^2-1}$$

Observe that the equation presented seems to be fractional then the procedure to follow to solve this type of equations is applied

Let's apply procedure 1 for your solution

$$\frac{3}{x-1} - \frac{6}{x^2-1} = 0$$

Let's perform the operation indicated between the two algebraic fractions

Let's break down the denominators into factors

Let's break down the denominators into factors

$$\frac{3}{x-1} - \frac{6}{(x+1)(x-1)} = 0$$

Let's determine the least common multiple between the denominators

$$m.c.m\big(x-1, (x+1)(x-1)\big) = (x+1)(x-1)$$

Performing the subtraction

$$\frac{3(x+1)-6}{(x+1)(x-1)} = 0$$

By carrying out the operations indicated in the numerator, we obtain

$$\frac{3x+3-6}{(x+1)(x-1)}=0$$

Reducing similar terms in the numerator

$$\frac{3x-3}{(x+1)(x-1)}=0$$

We have already obtained the fractional equation in its structure, that is, in the form $\frac{T_1}{T_2}=0$

Once this structure is obtained we verify if it is possible to simplify even more the algebraic fraction obtained

Decomposing the numerator into factors we obtain

$$\frac{3(x-1)}{(x+1)(x-1)}=0$$

And now notice that there is a factor in the numerator that can be simplified with a factor found in the denominator

$$\frac{3}{x+1}=0$$

Note that this simplified algebraic fraction is only zero if the numerator is zero and that is never possible since 3≠0 so it can be inferred that the initial equation has a null or empty solution and is represented as follows $S=\emptyset$ o $S=\{\,\}$

But suppose that when you got the fractional equation

$$\frac{3x-3}{(x+1)(x-1)}=0$$

You did not realize the possibility of breaking down the numerator into factors to reduce the algebraic fraction obtained to the maximum expression, no matter

Proceed as follows

When can only an algebraic fraction be zero?

Of course, only when the numerator is zero

Then let's equalize the numerator to zero

$3x-3=0$ and solve the linear equation obtained

From which it is obtained as a result $x = 1$

But it cannot yet be said that $x = 1$ is the solution of the fractional equation, it is necessary to check as indicated in procedure 1

It is very easy to see that it is not a solution because $x = 1$ does not belong to the domain of definition of the equation since the domain of the equation is $\{x \in R: x \neq \pm 1\}$

Note. To check the equation, you can also use the resource of substituting the possible solution in the initial equation and arrive at a contradiction

For example

When replacing $x = 1$ in the left member of the initial equation you have to $\frac{3}{1-1} = \frac{3}{0}$ and the division by zero is not defined so the conclusion is the same, the equation has empty solution.

Let's see another example using the procedure 1

Solve the following equation

$$\frac{5x-8}{x-1} = \frac{7x-4}{x+2}$$

The equation presented seems to be fractional, let's obtain its structure to corroborate that it is

$$\frac{5x-8}{x-1} - \frac{7x-4}{x+2} = 0$$

Let's solve the operation indicated in the left member of the equation

$$\frac{(5x+8)(x+2)-(7x-4)(x-1)}{(x-1)(x+2)} = 0$$

Removing the grouping signs from the numerator by performing the indicated operations

$$\frac{5x^2+10x-8x-16-(7x^2-7x-4x+4)}{(x-1)(x+2)} = 0$$

$$\frac{5x^2+10x-8x-16-7x^2+7x+4x-4}{(x-1)(x+2)} = 0$$

By reducing similar terms in the numerator, you have to

$$\frac{-2x^2+13x-20}{(x-1)(x+2)} = 0$$

Multiplying by -1 the equation to obtain the highest exponent variable of the positive numerator

$$\frac{2x^2 - 13x + 20}{(x-1)(x+2)} = 0$$

Let's try to factor the numerator, to see if it is possible to reduce to the maximum expression the algebraic fraction, obtained in the left member of the equation

$$\frac{(2x-5)(x-4)}{(x-1)(x+2)} = 0$$

Note that it does not admit any further simplification since there are no equal factors in the numerator and the denominator

Now you must ask yourself when an algebraic fraction could be zero?

Of course, when the numerator is zero, that is:

$$(2x-5)(x-4) = 0$$

And when is a product zero?

Very well, when at least one of its factors is zero

Therefore

$$2x - 5 = 0 \text{ ó } x - 4 = 0$$

By solving these linear equations, the following results are obtained

$$x = \frac{5}{2} \quad y \quad x = 4$$

These obtained values constitute the possible solutions of the initial equation; so, they must be verified, then it is necessary to check as required by procedure 1

But since the definition domain of the equation is $\{x \in R: x \neq 1, x \neq -2\}$ and possible solutions

$x = \frac{5}{2} \quad y \quad x = 4$ belong to the domain of the equation, it can be guaranteed that these finally are solutions of the equation

Therefore $S = \{\frac{5}{2}, 4\}$

Note that by having two solutions first he put the smaller one in the solution set.

Let's analyze the other way to check, that is, substituting the possible solutions in the initial equation.

As there are two possible solutions it is necessary to perform two checks

Check for x=4

$$\frac{5x-8}{x-1}=\frac{7x-4}{x+2}$$

Let's replace x=4 in the equation

$$\frac{5.4-8}{4-1}=\frac{7.4-4}{4+2}$$

$$\frac{20-8}{3}=\frac{28-4}{6}$$

$$\frac{12}{3}=\frac{24}{6}$$

From where you have to

$$4=4$$

Obtaining a true proposition therefore $x=4$ is solution of the equation

Let's check for $x=\frac{5}{2}$

The procedure is analogous to the previous one

Replaced by $x=\frac{5}{2}$ in the initial equation

$$\frac{5.\frac{5}{2}-8}{\frac{5}{2}-1}=\frac{7.\frac{5}{2}-4}{\frac{5}{2}+2}$$

$$\frac{\frac{25}{2}-8}{\frac{3}{2}}=\frac{\frac{35}{2}-4}{\frac{9}{2}}$$

$$\frac{\frac{9}{2}}{\frac{3}{2}}=\frac{\frac{27}{2}}{\frac{9}{2}}$$

$$\frac{9}{2}.\frac{2}{3}=\frac{27}{2}.\frac{2}{9}$$

From where you have to

$3 = 3$ and as this proposition is true then we can affirm that $x = \frac{5}{2}$ is solution of the given equation, therefore the set solution of the equation is $S = \{\frac{5}{2}, 4\}$

Let's see now another procedure to follow to solve fractional equations and that is the most used by teachers and students

Procedure 2

1-Find the lowest common multiple of the denominators

2-Multiply both members of the equation by the least common multiple of the denominators to eliminate the denominators of the equation and transformed into a simpler equation.

3-Solve the new equation obtained.

4-Check the value obtained

5-Write the solution or set solution

Let's see an example

Solve the following equation

$$\frac{x+1}{x+2} = 2$$

As there is only one denominator, the minimum common multiple (lowest common multiple.) coincides with the only denominator that exists, therefore the lowest common multiple. is $x + 2$

Let's multiply both members by $x + 2$

$x + 1 = 2(x + 2)$

By removing grouping signs you get

$x + 1 = 2x + 4$

Now we have to solve the originated equation that seems to be linear, then we apply the procedure to follow to solve a linear equation

$0 = 2x + 4 - x - 1$

$0 = x + 3$

$x = 3$

Once the possible solution is obtained, it is necessary to check

You know that to check this type of equation you can use any of the forms above

Since the domain of the equation is $\{x \in R: x \neq -2\}$ and x=3 belongs to it, then it can be guaranteed that the possible solution is solution of the equation

Therefore $S = \{3\}$

We suggest that you perform the check by substituting the possible solution in the initial equation. Also, solve the equation using procedure 1 and compare the results

Let's see another example

Let's solve the following equation

$$\frac{1}{x} = \frac{3x-1}{x^2-x} - \frac{2}{x-1}$$

Let us determine the lowest common multiple among the denominators, but for this as it appears more than one denominator is necessary to decompose into factors all the denominators that appear whenever possible.

$$\frac{1}{x} = \frac{3x-1}{(x+1)(x-1)} - \frac{2}{x-1}$$

Remember that to determine the lowest common multiple among several algebraic expressions, the product of the common factors is written and the uncommon ones and of the common ones the one with the highest exponent is taken.

In this case, the lowest common multiple would be (x , $(x+1)(x-1), x-1) = x(x+1)(x-1)$

By multiplying both members by $(x+1)(x-1)$ we obtain

$$\frac{1}{x}x(x+1)(x-1) = \frac{3x-1}{(x+1)(x-1)}x(x+1)(x-1) - \frac{2}{x-1}x(x+1)(x-1)$$

Carrying out the indicated multiplication you have to

$$(x+1)(x-1) = (3x-1)x - 2x(x+1)$$

Note that when the whole equation was multiplied by the lowest common multiple all the denominators were eliminated, which you should always achieve when you are solving a fractional equation using procedure 2

Now let's solve the equation that originated

By simple inspection and very a priori seems to be a linear equation to observe that the largest exponent of the variable is 1 therefore apply the procedure to solve an equation of this type

Let's eliminate the grouping signs through the operations indicated

$$x^2 - 1 = 3x^2 - x - 2x^2 - 2$$

Notice now that the transformed equation seems to be quadratic because the highest exponent variable is 2, then following this logic of analysis, Let's use the procedure to solve a quadratic equation, which suggests transposing all the term to one member getting to zero in one of the members of the equation.

$$0 = 3x^2 - x - 2x^2 - 2 - x^2 + 1$$

But reducing similar terms in the right member to obtain the form of the second-degree equation results in

$$0 = -x - 1$$

And as a surprise we take away that the equation at the end obtained is linear since it has the structure of a linear equation

Clearing the variable, we have to $x = -1$ which is the possible solution to the equation

But in the fractional equations the verification is mandatory, so let's check.

The domain of the given fractional equation is $\{x \in R: x \neq 0, x \neq 1, x \neq -1\}$ y como and as the possible solution does not belong to the domain of the equation then it can be stated that the equation has an empty solution so the whole solution can be written as follows $S = \emptyset$

It should be made clear that in the procedure of solving these equations the denominators must be eliminated. To do this, both members are multiplied by the least common multiple, and when multiplying, the values of the variable that make zero the least common multiple can be solutions of the transformed equation, but not of the original one, since they are precisely the values that are not in their definition domain.

Therefore, it should be checked in the original equation or instead of doing the check, it should be analyzed whether the solutions found belong or not to the definition domain of the equation.

1.3.2 Exercises Proposal

Solve the following equations

a- $x+\frac{3}{x}=\frac{x^2+3+2x}{x}$

b- $\frac{15}{x}-\frac{11x+5}{x^2}=1$

c- $\frac{1}{3x-3}+\frac{1}{4x+4}=\frac{1}{12x-12}$

d- $\frac{1}{(x-1)^2}-\frac{3}{2x-2}=-\frac{3}{2x+2}$

e- $\frac{2}{3}-\frac{6x^2}{9x^2-1}=\frac{2}{3x-1}$

f- $\frac{3x-4}{x^2-3x}-\frac{2}{x-3}=\frac{1}{x}$

g- $\frac{1}{x^2+3x-28}-\frac{1}{x^2+10x+25}=\frac{3}{x^2+x-20}$

h- $\frac{2(x+2)}{x-2}-\frac{3(x-2)}{2x+3}=\frac{x^2-52}{2x^2-x-6}$

i- $\frac{x-2}{x^2+8x+7}=\frac{2x-5}{x^2-49}-\frac{x-2}{x^2-6x-7}$

j- $\frac{x}{x-3}-\frac{3}{x+5}=\frac{24}{x^2+2x-15}$

k- $\frac{x-1}{x+1}=\frac{2x+9}{x+3}-\frac{x+1}{x-1}$

l- $\frac{x-2}{x^2-x-6}=\frac{x}{x^2-4}+\frac{3}{2x+4}$

m- $\frac{x+7}{3x-3}-\frac{x+3}{x+1}=\frac{4}{3}$

n- $\frac{x+2}{x-2}-\frac{2}{x-5}=\frac{2x}{x^2-7x+10}$

o- $\frac{x^2+x+3}{x^2-x+3}=\frac{2x+5}{2x+7}$

p- $\frac{2x^2+11x+5}{x^3+5x^2-4x-20}.(x^2-4)+x^2=0$

q- $\frac{x^2+5x}{x^2+x-12}+\frac{x}{x-3}=1$

1.4 Equations containing a variable under the sign of module

Definition. It is called modular equation to that type of equation where the variable is under the sign of module.

For example

$|4x-6|=1$, $|2x+3|+|x-1|=4$

1.4.1 Procedures for solving modular equations

Squared lift method.

This method is very suggestive when you can guarantee that both members of the equation are positive.

Example 1

Let's solve the following equation

$|2x-3|=|x+7|$, as you can be seen both members of the equation are positive.

$|2x-3|^2=|x+7|^2$, from where you have to $(2x-3)^2=(x+7)^2$, When solving this equation you have the following solutions $x_1=10$ y $x_2=-\frac{4}{3}$

Example 2

$|x-2|=1$, Note that it can be guaranteed that both members of the equation are positive for the entire set of allowable values of the variable.

Squaring both members, we obtain that:

$(x-2)^2=1^2$, solving the indicated operations $x^2-4x+4=1$, resulting in a quadratic equation

$x^2-4x+3=0$

$(x-3)(x-1)=0$

$x=3$

$x=1$

So the whole solution S= { 1, 3 }

Interval splitting method.

This method is suggested when it cannot be guaranteed a priori that both members of the equation are positive and usually when more than one module sign appears in the equation.

Procedure

1- Determine the values that cancel each of the modules involved in the equation.

2- Represent the determined values on the numerical line (this will partition the line into intervals)

3- Analyze the behavior of the equation in each of the originated partitions

4- Check if the possible solution belongs to the analysis interval

5- Give a joint solution.

Example

Let's solve the following equation

$$|3-x|-|x+2|=5$$

Note that we cannot guarantee a priori that the left member of the equation is positive for the whole set of admissible values of the variable.

To reduce this problem to one already known to us, it could be done $|3-x|=5+|x+2|$ and with this operation we already guarantee that both members are positive and therefore we can use the method of squaring, which we leave to the reader, but this method for this particular case is cumbersome, because of the amount of modules that appear.

Let's apply the interval partition method.

Let's determine the zeros of each one of the modules that intervene in the initial equation.

$3-x=0$ y $x+2=0$, from where it is obtained that $x=3$ y $x=-2$

These two values cause the line to be divided into three intervals $]-\infty,-2[$, $[-2,3]$ y $]3,+\infty[$.

Let's analyze in the first interval how the equation would look like $3-x-(-(x+2))=5$

Solving the originated equation you have to

$3-x+x+2=0$, from which we have the identity 5=5 which means that all the values belonging to this interval constitute solution of the initial equation.

Let's carry out the analysis in the second interval, that is $[-2,3]$

The equation would be expressed as follows:

$3-x-(x+2)=5$, Solving the equation results in $3-x-x-2=5$

$3-2x-2=5$

$-2x=4$

$x=-2$

As the solution of the equation belongs to the analysis interval then $x=-2$ is solution of the equation.

Let's analyze the behavior of the equation in the $]3,+\infty[$

The equation would be as follows

$-3+x-x-2=5$, from which the contradiction -5=5 is obtained, which means that no value of this interval constitutes solution of the initial equation.

Therefore the set solution of the equation is S=$]-\infty,-2[\cup${ -2 }=$]-\infty,-2]$

Example 3

Solve the following equation

$|x-2|+|x-1|=x-3$

Let's determine the zeros of each module

$x-2=0$, $x-1=0$, obtaining the following values $x=2$ y $x=1$, originating in the straight three partitions $]-\infty,1[$, $[1,2]$ y $]2,+\infty[$

Let's analyze the equation in the first interval

$-(x-2)-(x-1)=x-3$

$-x+2-x+1=x-3$

$-2x+3=x-3$

$-3x=-6$

$x=2$

But as $x=2$ does not belong to the analysis interval is not a solution of the initial equation.

Let's make the analysis of the equation in the interval $[1,2]$

$-(x-2)+x-1=x-3$

$-x+2+x-1=x-3$

$x=4$

But $x=4$ does not belong to the analysis interval either, so it is not a solution of the equation.

Let's analyze the last partition.

The equation would be as follows

$x-2+x-1=x-3$

$x=0$

And let's observe that this value does not belong to the analysis interval either, so it can be decided that the equation has no solution, that is S={ }

1.4.2 Exercises Proposal

1-Solve the following equations

a- $|4x-7|=7-4x$

b- $|3x-5|=5-3x$

c- $4x-8=x|x-2|$

d- $|x|+x^2=0$

e- $\frac{7x+4}{5}-x=\frac{|3x-5|}{2}$

f- $|x|+|x+1|=1$

g- $|x+1|+|x+2|=2$

h- $|2x-1|-|3-x|=|x-4|$

i- $|x-2|+|x-3|+|2x-8|=0$

j- $|x|+2|x+1|-3|x-3|=0$

k- $|x+1|-|x|+3|x-1|-2|x-2|=|x+2|$

1.5 Irrational or radical equations

Definition. The irrational equations, or equations with radicals, are those that have the unknown under the sign of radical or under the sign of elevation to a fractional power.

Examples

a-) $\sqrt{3x+4}-3x=-2$ b-) $2\sqrt{x}+5=7$ c- $3\sqrt{2x}+4\sqrt{3x+2}=5\sqrt{x+3}$

The following examples are not equations with radicals

a−)$x+\sqrt{5}=3x$ is not an equation with radicals because there are no variables under the sign of the radical, as you will see this is an equation that when transformed is a linear equation.

b-)$\sqrt{2x-1}$ is not an equation with radicals because although the variable is under the sign of the radical, we are not in the presence of an equation for lack of equality.

1.5.1 Procedures for solving equations with radicals

Method of raising both members of the equation to the same power.

1- A radical is isolated in one of the two members, passing to the other member the rest of the terms, even if they also have radicals.

2- Both members of the equation are elevated to the power that indicates the index of the isolated radical.

3- The equation obtained is solved.

4-It is checked whether the solutions obtained satisfy the initial equation. It must be taken into account that when squaring, or squaring an equation to an even power, strange solutions can be introduced since squaring is not an equivalent transformation in the domain of real numbers.

Example 1

Solve the following equation

$\sqrt{2x-5}=7$

Note that the equation is with radicals

As you see the radical is isolated which means being alone in one member of the equation

Therefore, you must go on to the next step of the procedure that consists of squaring both members of the equation (It is important that you know that it squares because the index of the radical is 2, if it were a cubic root it must be squared to 3 and so on)

$$\sqrt{2x-5}=7 \qquad /(\,)^2$$

$$\left(\sqrt{2x-5}\right)^2=49$$

$$2x-5=49$$

$$2x = 54$$

$$x = 27$$

Remember that any irrational or radical equation must be checked because when the equation is raised to an even power, the equation is transformed into another, so in some cases its solution does not satisfy the original equation.

Let's check the original equation:

$$\sqrt{2\cdot 27-5}=7$$

$$\sqrt{54-5}=7$$

$$\sqrt{49}=7$$

$$7 = 7$$

Therefore x = 27 satisfies the equation, that is, it is its root or solution.

Let's write then the solution set of the equation $S=\{27\}$

Another way to check the given equation is to determine the domain of definition of the equation, and if the possible solution belongs to this one then it will be solution of the equation, if on the contrary it does not belong to the domain then it will not be solution of this one.

Let us determine the domain of definition of the equation

$2x-5\geq 0$ (since the index of the radical is even, it is necessary that the radicand be greater or equal to zero), from which it is obtained $x\geq\frac{5}{2}$, therefore the domain of the equation is $\{x \in R : x \geq \frac{5}{2}\}$ and since 27 belongs to this then it is solution of the equation

Example 2

Solve

$\sqrt{x+5} + \sqrt{x+2} = 6$

$\sqrt{x+5} + \sqrt{x+2} = 6$ Here it is convenient to isolate a radical as suggested by the procedure:

$\sqrt{x+5} = 6 - \sqrt{x+2}$ note that there is only one radical left in the left member of the equation so it is isolated

$\left(\sqrt{x+5}\right)^2 = \left(6 - \sqrt{x+2}\right)^2$ now let's raise both members of the equation squared in order to eliminate a radical

$x + 5 = 36 - 12\sqrt{x+2} + x + 2$ the right member was developed by squaring a binomial, that is using $(a-b)^2 = a^2 - 2ab + b^2$ where $a = 6\ y\ b = \sqrt{x+2}$

But notice that the originated equation is again an equation with radicals so again we have to apply the procedure to solve an equation of this type.

Let us clear then the radical

$12\sqrt{x+2} = 36 + x + 2 - x - 5$

By reducing similar terms in the right member of the equation and squaring both members you have to

$$\left(12\sqrt{x+2}\right)^2 = (33)^2$$

$$144(x+2) = 1089$$

$$144x + 288 = 1089$$

$$x = \frac{89}{16}$$

Let's check using this value in the original equation

$$\sqrt{\frac{89}{16}+5}+\sqrt{\frac{89}{16}+2}=6$$

$$\sqrt{\frac{169}{16}}+\sqrt{\frac{121}{16}}=6$$

$$\frac{13}{4}+\frac{11}{4}=6$$

$$\frac{24}{4}=6$$

and we obtain 6=6, Therefore $\frac{89}{16}$ is its root or solution

Therefore $S=\{\frac{89}{16}\}$

Method for introducing new variables or auxiliary variables

In certain occasions in the solution of an irrational equation the process of isolating the radical and the later elevation to the power indicated by the index of the isolated radical, lead to a voluminous equation somewhat complicated to solve, for which it is necessary to resort to other procedures that allow a greater simplification of the work.

For example.

Solve the following equation.

$$x^2+3-\sqrt{2x^2-3x+2}=1{,}5(x+4)$$

If we multiply both members of the equation by 2, we get

$2x^2+6-2\sqrt{2x^2-3x+2}=3x+12$ (for the convenience of all numbers being integers).

Below you have

$2x^2-3x+2-2\sqrt{2x^2-3x+2}-8=0$, by doing $y=\sqrt{2x^2-3x+2}$, is obtained

$y^2-2y-8=0$, from where $y=4$, $y=-2$

By returning to the original variable of the equation, the following pair of equations is obtained:

$\sqrt{2x^2-3x+2}=4$, y $\sqrt{2x^2-3x+2}=-2$

From the first equation it is obtained that $x=\frac{7}{2}$ y $x=-2$, of the second equation are not rooted because for the set of admissible values of the variable, the left member is always positive, while the right member is always negative. When checking, one has that the possible solutions obtained constitute solutions of the original equation.

Artificial resolution procedures
Solve the following equation:

$$\sqrt{2x^2+3x+5}+\sqrt{2x^2-3x+5}=3x \qquad \textbf{(1)}$$

Let's multiply both members of the equation by the expression

k(x)= $\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5}$, which is the conjugation of the expression $\sqrt{2x^2+3x+5}+\sqrt{2x^2-3x+5}$, by multiplying both members of the equation by the expression k(x) is obtained $(2x^2+3x+5)-(2x^2-3x+5)=3x\left(\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5}\right)$

From where one has by reducing similar terms that $6x=3x\left(\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5}\right)$, subtracting $6x$ in both members is obtained $3x\left(\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5}\right)-6x=0$, extracting common factor 3x you have $3x\left(\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5}-2\right)=0$

A product is zero when at least one of the factors is zero, therefore $3x=0$, ó

$$\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5}-2=0$$

$$\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5}=2 \qquad \textbf{(2)}$$

Adding (1) and (2), we arrive at the following equation $2\sqrt{2x^2+3x+5}=3x+2$

Solving this equation using the squared elevation method results in $8x^2+12x+20=9x^2+12x+4$. Reducing similar terms results in $x^2=16$, where x=4, x=-4, Checking the equation which we leave to the reader, it is concluded that only x=2 is solution of the equation, being the only root of the equation.

1.5.2 Exercises Proposal

1.- Solve the following irrational equations:

a) $\sqrt{x+5}+\sqrt{3}=\sqrt{x+7}$

b) $2\sqrt{\sqrt{3x}}=4$

c) $\sqrt[3]{2\sqrt{3x+4}}=2$

d) $\frac{3}{4}\sqrt{\sqrt{x}}=1$

e) $\sqrt{\sqrt{x+1}}=1$

f) $\sqrt[5]{32\sqrt[3]{3\sqrt{x}}}=2$

g) $\frac{1}{2}\sqrt[4]{2\sqrt[3]{x+1}}=1$

h) $\sqrt[3]{x^3+3x^2}=x+1$

i) $\sqrt[3]{x^3+6x^2+5x+8}=x+2$

j) $\frac{1}{2}-\frac{1}{2\sqrt[3]{x}}=\frac{2}{\sqrt[3]{x}}$

k) $\frac{1}{\sqrt{x+4}}=\frac{\sqrt{x-4}}{3}$

l) $2\sqrt{5+x}+\sqrt{9-3x}=\sqrt{41-3x}$

2.- Put forward the equation and solve the following problems:

a) The area of an equilateral triangle is $9\sqrt{3}$ m^2. Calculates the perimeter and height measurement.

b) The volume of a cube measures 1728 m^3. Calculate the measure of the diagonal of one of its faces and the measure of the diagonal of the cube.

c) If the square root of a number is increased by two, the result is 5. What is the number?

d) The volume of a sphere measures 36 π m^3. Calculate the size of your radius.

e) Calculate the perimeter of a rhombus whose diagonals measure 6 m and 8 m, respectively.

f) The dimensions of the sides of a triangle are 26 m, 24 m and 10 m, respectively. Calculate your area.

g) The volume of a straight cone measures 245 π cm^3. How big is your radius if your height is 15 cm?

h) The area of an equilateral triangle is 100$\sqrt{3}$ m^2. Indicate the size of the area of the square that has the height of the triangle on its side.

i) The area of a square is 8 m^2. Calculate the size of the area of the square that has the diagonal of the square as its side.

j) Calculate the measure of the area of the cube that has as edge the diagonal of a cube whose volume measures 729 m^3.

k) Calculate the area of an equilateral triangle whose side has the same measurement as the side of a square in which its diagonal measures $7\sqrt{2}$ m.

Chapter 2. Transcendent equations

Definition. An equation is called transcendent, if at least one of the terms is not algebraic.

Example of transcendent equations.

a) $3^x = 4^{x-2} \cdot 2^x$

c) $senx = \cos^2 x - \frac{1}{4}$

b) $2\log_5(3x-1) - \log_5(12x+1) = 0$

d) $x\cos x = senx$

In some cases the resolution of transcendent equations is reduced to the resolution of algebraic equations; and in many occasions they can only be solved approximately.

In this chapter we will only deal with the exponential, logarithmic and trigonometric transcendent equations.

2.1 Exponential equations

Definition. An equation is called exponential if the variable is part only of the exponents of the powers for certain bases.

Example

a) $2^{3x+1} = 4$ b) $\left(\frac{1}{3}\right)^{x^2-2} = 3^{x-2}$ c) $3^{2x} + 4.3^{x+1} = 5$ d) $4^{\sqrt{2x-1}}.2^3 = 16$

Exponential equations are used to solve multiple practical applications such as population growth, radioactive decay, and compound interest.

The following example involves the growth of populations, which tend to grow exponentially and at different rates. A convenient and easy way to understand the measure of growth rate is the doubling time (this is the time it takes for a population to double). Over short periods, the doubling time growth model is often used to model population growth

$P = P_0 2^{\frac{t}{d}}$, where P is the population at time t, P_0 population at time t = 0, and d duplication time.

Note that when t = d you have to $P = P_0 2^{\frac{d}{d}} = P_0 2$, and the population is double the original one.

Let's see a practical example

Mexico has a population of approximately 100 million people, and it is estimated that it will have doubled in 21 years. If it continues to grow at the same rate, what will the population be in 15 years from now?

Let's use the model of doubling time growth $= P_0 2^{\frac{t}{d}}$, replacing $P_0 = 100 \; and \; d = 21$ we get $P = 100(2^{\frac{15}{21}}) \approx 164$ million people

To solve an exponential equation, it is important to take into account the following definition and properties of the powers.

Definition. For every positive real number and every real number, there is only one real number $b > 0$ such that $a^c > 0$ and $a^c = b$

4.1.1 The powers have the following properties:

Si $a > 0$, $b > 0$ y $m, n \in R$

$a^0 = 1$ $\quad a^1 = a$ $\quad a^{-1} = \frac{1}{a}$ $\quad a^n . a^m = a^{n+m}$ $\quad a^n : a^m = a^{n-m}$

$(a^m)^n = a^{m.n}$ $\quad a^n . b^n = (a.b)^n$ $\quad a^n : b^n = (a:b)^n$ $\quad \sqrt[n]{a^m} = a^{\frac{m}{n}}$

2.1.1 Procedures for solving exponential equations

Procedure 1

Reduction to shape $a^{T(x)} = a^{M(x)}, (a \neq 1)$. The following instructions can be followed:

1.Obtain an equality between two powers with the condition that they have the same base using the properties of the powers, that is to say $a^{T(x)} = a^{M(x)}$.

2. Equalize the exponents.

3.Solve the equation that originates as a result of equalizing the exponents.

4.Perform the verification of the equation if the originating equation requires verification.

5.Express the solution.

Example

Solve the following equation:

$$2^{\sqrt{x^2+5x}}.4 = 2^x \quad \text{(I)}$$

Note that we are in the presence of an exponential equation for the reason that the variable is in the exponent of a power. From here it is obtained,

$2^{\sqrt{x^2+5x}}.2^2 = 2^x$ The same is true for the other powers.

$2^{\sqrt{x^2+5x}+2} = 2^x$, applying properties of the power.

$\sqrt{x^2+5x}+2 = x$ (II), equalizing the exponents.

As it can be observed, the equation obtained is an equation with radicals, so for its solution, procedures studied previously must be used.

$$\sqrt{x^2+5x} = x-2$$
$$x^2+5x = (x-2)^2$$
$$x^2+5x = x^2-4x+4$$
$$9x = 4$$
$$x = \frac{4}{9}$$

Therefore it $x=\frac{4}{9}$ constitutes the possible solution of the equation, but it is necessary to perform the verification since the equation originated with radicals, and this is necessary to check since the squared elevation is not always an equivalent transformation and strange solutions can be introduced.

Let's check in (I).

The left member is $2^{\sqrt{x^2+5x}}.4$, substituting $x=\frac{4}{9}$ in it we have to:

$$2^{\sqrt{\left(\frac{4}{9}\right)^2+5\left(\frac{4}{9}\right)}}.4=2^{\sqrt{\frac{16}{18}+\frac{20}{9}}}\cdot 4=2^{\sqrt{\frac{196}{81}}}\cdot 4=2^{\frac{14}{9}}\cdot 4=2^{\frac{14}{9}}\cdot 2^2=2^{\frac{32}{9}}$$

$x\geq 2$ and since the possible solution does not meet this condition by being $\frac{4}{9}\leq 2$, it can be decided that it $\frac{4}{9}$ is not a solution of the original equation.

Procedure 2

Change of variables. The following instructions can be followed:

1-Transform all the powers until achieving equal powers

2-Realize change of variables

3-Solve the algebraic equation obtained.

4-Solve the simple exponential equations originated by changing variables.

5-Express the solution.

Example

Solve the following equation*:* $2\cdot 2^{2x}=3-5\cdot 2^x$

Note that it is impossible to obtain only one power in each member of the equation, because it is in the presence of power sums that are not similar and therefore; the properties of power multiplication and division cannot be applied.

Applying power property of power, we obtain

$$2\cdot\left(2^x\right)^2 = 3-5\cdot 2^x \text{ (I)},$$

By making a change in (I) variables $y = 2^x$ (II), the algebraic equation is obtained $2y^2 = 3-5y$,

so it is suggested to solve in this case the quadratic equation obtained.
$2y^2 + 5y - 3 = 0$
$(y+3)(2y-1) = 0$

$$y+3=0 \quad \text{ó} \quad 2y-1=0$$
$$y=-3 \qquad\qquad y=\frac{1}{2}$$

Substituting in the initial variable from the relation (II) the following exponential equations are obtained: $-3 = 2^x$ (III) y $\frac{1}{2} = 2^x$ (IV)

In equation III the situation is impossible. In equation IV we obtain $2^{-1} = 2^x$ and therefore $x = -1$.

The solution to the equation will be $S = \{-1\}$. In this case it was not necessary to develop the verification since all the transformations performed were equivalent.

Note: to clear an unknown that is in the exponent of a power, logarithms are taken whose base is the base of the power. The solution of the equation will be

$$a^x = b \Rightarrow \log_a a^x = \log_a b \Rightarrow x = \log_a b$$

2.1.2 Exercises Proposal

1. Determine the solution to the following equations:

$a) 2^{\frac{x}{3}+2} = 8$

$b) 36^{x+3} = 216$

$c) \left(4^{x-2}\right)^{x} = 8^{x}$

$d) \left(5^{x-2}\right)^{x-3} = 1$

$e) 7^{3x^2+x} = 49$

$f) 4^{\frac{x-1}{2}} = 8^{x^2-1}$

$g) 9^{-3x} = \left(\frac{1}{27}\right)^{x+3}$

$h) 12^{\frac{1-x}{x}} = \frac{12}{12^{x}}$

$i) \left(3^{x^2-x}\right)^{\sqrt{2}} = 9^{\sqrt{2}}$

1. For which values of $x \in R$, is fulfilled:

$a) 10 \cdot 2^{x^2-4} = 320$

$b) 2^{2x} - 9 \cdot 2^{x} + 8 = 0$

$c) 3^{2x} - 8 \cdot 3^{x} - 9 = 0$

2. Dice $A = \frac{2x^2 - x - 3}{x}$ and $B = \frac{3x}{6x-9}$, solves $2^{A \cdot B} = 16$.

3. Knowing that:

$E = \frac{2-x}{x^2 - 4x + 4}$, $F = \frac{6x^2 - 18x}{2x^2 - 4x}$ y $G = \frac{x^2 - 6x + 8}{3x^2 - 4x - 20}$

4.Determines the value x of whether $3^{(E+F)G} = 81$

5.Find the set solution of the following equations:

$a) 1 = \sqrt{1 - \sqrt{4^{x+1} - 7 \cdot 16^{x} + 2^{x}}}$

$b) 9^{\frac{\sqrt{x}}{2}} \cdot 3^{\sqrt{x+5}} = \left(\sqrt{3}\right)^{2x+2}$

$c) \left(\frac{1}{2}\right)^{\sqrt{x^6 - 2x^3 + 1}} = 2^{x-1}$

$d) \left(\sqrt[3]{2}\right)^{\frac{x+1}{x-2}} : 4^{\frac{x-1}{2x+4}} = 2^{\frac{28-x^2}{6x^2-24}}$

2.2. Logarithmic equations

Definition. An equation is called logarithmic if the variable is affected by a logarithm, and it is in the base, in the argument or in both at the same time.

Example

a) $\log(x+1)=3$ b) $\log_2(2x-1)=\log_2(x^2=1)$ c) $\log_{x+2}4=2$

d) $\log_x(3x+6)=2$ e) $\log_2(x+4)=\log_4 2x$

Let's analyze the following applications:

-In 1935, the American seismologist Charles Richter developed a logarithmic scale that bears his name and is widely used to calculate the magnitude of the intensity of an earthquake using the equation: $M=\frac{2}{3}\log\frac{E}{E_0}$

where E it is the energy released by the earthquake, measured in joules, and E_0 is the energy released by a very mild earthquake that has been standardized as: $E_0=10^{4.40}\,joules$.

- The theory of flight of a rocket is used in advanced mathematics and physics to show that the speed of a rocket when it goes off (when the fuel is exhausted) is given by $v=c\ln\frac{W_t}{W_b}$

where c is the engine exhaust speed, W_t is the starting weight (fuel, structure and payload), and W_b is the consumed weight (structure and payload)

- The hydrogen ion concentration of a substance is related to its acidity and basicity. Because hydrogen ion concentrations vary over a very wide range, a tablet scale is used pH, which is defined:

$$pH=-\log[H^+]$$

where $\left[H^{+}\right]$ is the concentration of the hydrogen ion, in moles per liter? As can be pointed out, each of the phenomena addressed were resolved using common and natural logarithms.

To solve a logarithmic equation, it is important to take into account the following definition and properties.

Definition. Given two real numbers a,b ($a > 0$, $a \neq 1$ y $b > 0$) it is called logarithm on the basis a of b and denotes $\log_a b$ the number x that satisfies the equation $a^x = b$.

Properties of the logarithms

Whether x and y are real numbers (x > 0, y > 0), a real number (a > 0, a≠ 1), n∈ N (n > 1)

1) $\log_a x^k = k \log x \quad (k \in \mathrm{R})$ ∈R (fundamental logarithmic identity)

2) $\log_a a^b = b,\ b \in R$ (fundamental logarithmic identity)

3) $\log_a (xy) = \log_a x + \log_a y$

4) $\log_a \frac{x}{y} = \log_a x - \log_a y$

5) $\log_a x^k = k \log x \quad (k \in \mathrm{R})$

6) $\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$

7) $\log_{a^k} x = \frac{1}{k} \log_a x, \quad k \neq 0$

Expressions for the change of bases

Let the real numbers a, b, c and x be such that a > 0, a≠ 1, b > 0, c > 0, c ≠ 1 and x ≠ 0, then:

1) $\log_a b = \log_a c \,.\log_c b$

2) $\log_a b = \frac{1}{\log_b a} \quad b \neq 1$

2.2.1 Procedures for solving logarithmic equations

Procedure 1:

1- Obtain all logarithms with the same base

2- Transpose all logarithms for one member and independent terms for the other.

3- Obtain only one logarithm by applying properties

4- Apply logarithm definition

5- Solve the equation that originates

6- Checking

7- Express the solution.

Example:

Solve the following equation

$$\log(x-2)=\log(9x-18)+1$$

As we can see, all the logarithms involved have the same basis. Transposing all the logarithms to the left member of the equation we obtain that.

$$\log(x-2)-\log(9x-18)=1$$
$$\log\left(\frac{x-2}{9x-18}\right)=1$$
$$10^{1}=\frac{x-2}{9x-18}$$

As you can see the equation obtained can be made the following transformations:

$$10(9x-18)=x-2$$
$$90x-180=x-2$$
$$89x=178$$
$$x=2$$

Therefore, the possible solution to the equation is $x=2$

Checking we obtain that it does $x=2$ not belong to the definition domain of the initial equation so the possible solution does not constitute solution of the equation, then.

Therefore, the possible solution of the equation is

Checking we obtain that it does not belong to the definition domain of the initial equation so the possible solution does not constitute solution of the equation, then $S=\Phi$.

Procedure 2. Reducing the equation to the form: $\log_a T(x)=\log_a M(x)$

1.Obtain only one logarithm in each member of the equation applying properties, with the characteristic having the same base.

2. Equalize the arguments.

3.Solve the equation that originates.

4.To make the verification

5.Express the solution.

Example

Solve the following equation

$$\log(x-2)+\log(x-3)=1-\log 5$$
$$\log(x-2)+\log(x-3)=\log 10-\log 5$$
$$\log(x-2)(x-3)=\log\frac{10}{5}$$

Matching the arguments, we get the following equation.

$$(x-2)(x-3)=\frac{10}{5}$$
$$(x-2)(x-3)=2$$
$$x^2-5x+6=2$$
$$x^2-5x+4=0$$
$$(x-4)(x-1)=0$$

$x-4=0, x-1=0$

From where it has to $x=4, x=1$, are the possible solutions of the given equation. Let's do the check.

The possible solution $x = 1$, is not solution of the equation because it does not belong to the definition domain of the equation.

The opposite happens with $x = 4$, that if it is solution because it belongs to the domain of definition of the equation.

We suggest to the reader to make the verification of the equation replacing the values obtained in both members of the equation and verify the results previously stated.

Then, the solution of the equation is $S = \{4\}$.

Procedure 3 Using change of variables:

1- Elimination of grouping signs and reduction of similar terms

2- Changing variables

3- Solve the equation that originates

4- Return to the initial variable and solve the simple logarithmic equations that originate

5- Perform the check in the initial equation

6- Express the solution.

Example

Solve the following equation

$$(1 - \log x)\log x = -2$$

Eliminating the grouping signs we have to:

$$\log x - \log^2 x = -2$$

Making the change of variables $y = \log x$ we obtain the following rational equation

$y - y^2 = -2$ Obviously, a second-degree equation $y^2 - y - 2 = 0$

Decomposing into factors you have to

$$(y - 2)(y + 1) = 0$$

from where $y = 2$, $y = -1$

Returning to the initial variable we obtain that

$$\log x = 2 \text{ and } \log x = -1$$

By solving the simple logarithmic equations originated we obtain

$\log x = 2 \Rightarrow x = 10^2$ and of $\log x = -1 \Rightarrow x = \frac{1}{10}$

These are the possible solutions of the initial equation.

When performing the check, which we leave to the reader, the solution of the initial equation must be.

These being the possible solutions to the initial equation $S = \left\{ \frac{1}{10}, 10^2 \right\}$.

2.2.2 Exercises Proposal

1. Determine the set solution of the following equations:

$a) \log_2(x+3) + \log_2(x-4) = 3$
$b) 2\log_{10} x = \log_{10} 4 + \log_{10} 3x$
$c) \log_{\sqrt{2}}(2x+1) - \log_{\sqrt{2}}(5x+2) = 2$
$d) \log_5(x^2-100) - \log_5(x-10) = 2$
$e) \log_2 x = 3 - \log_2 7$
$f) \log_{10} x^2 = \log_{10}\left(x + \frac{11}{10}\right) + 1$
$g) \log_2 x + \log_8 x = 8$
$h) \log_x 5 + \log_{25} x = \frac{3}{2}$
$i) \log_2(x^2+2) = \log_{\frac{1}{2}}(x^2-2)$

.2. Find the values that satisfy each of the following equations::

$a) \log_2(9 - 2^x) = 25^{\log_5 \sqrt{3-x}}$
$b) \log_x(2x^{x-2} - 1) + \log_x x^4 = 2x$
$c) 0{,}5 \cdot \log_7(2x^2 - 4) - \frac{1}{2} = \log_7(x-2)$
$d) \log_3(9^x - 2 \cdot 3^x - 2) + 1 = x$
$e) 0{,}4^{\log^2 x + 1} = 6{,}25^{2 - \log x^3}$

3. Be and numbers in the interval $(0;1)$, with the property that there is a positive number a different from 1, such that $\log_x a + \log_y a = 4\log_{xy} a$. It proves that $x = y$.

4. If the energy released by one earthquake is 1000 times that of another, how much larger on the Richter scale is the reading of the larger one compared to the reading of the smaller one?

5.An optical instrument is needed to observe the stars beyond the sixth magnitude, the limit of ordinary vision. However, optical instruments still have their limitations. The limiting magnitude L, of any optical telescope with a lens diameter D, in inches, is given by $L = 8.8 + 5.1\log D$

a) Find the magnitude limit for a 6-inch home reflector telescope.
b) Find the diameter of a lens that would have a limiting magnitude of 20.6.
Note: $1in \approx 2.54cm$

2.3. Trigonometric equations

Definition. A trigonometric equation is an equation where the variable appears in the argument of a trigonometric function.

Example

a) $senx + \cos 2x = 5$ b) $\tan x - sen2x = 1$ c) $2x + sen2 = 3$

Examples a) and b) are trigonometric equations, however c) is not because the variable is not in the argument of any trigonometric function.

Trigonometric equations have a wide use in practice, among which we can highlight Architecture, Air Safety, Engineering, Physics, Astronomy, in cost analysis, etc.

Example

The pole of an electric railroad is 5.20m high and is subjected to a horizontal tension force of 1020kgf, originated by the cable. The pole is fixed to the ground by a turnbuckle. Calculate the tensile force acting on the turnbuckle and the load on the post foundation if the weight of the post is known to be 800kgf and the distance from the foot of the post to the turnbuckle on the ground is 7.00m.

The examples a) and b) are trigonometric equations, however c) is not because the variable is not in the argument of some trigonometric function.

$tan \propto = \frac{7,00m}{5,20m} = 1,35$ from where $\propto = 53,5°$

We denote by x the modulus of tension force then x.sen∝=1020kgf

$$x = \frac{1020kgf}{sen53,5°} = \frac{1020kgf}{0,804} = 1270kgf$$

The tensile force acting on the tensioner is approximately 1270kgf

Let us denote by y and the load of the post y foundation and by z, the modulus of the component of the tensile force that influences the load, then y = z + F, z = 1020 kgf.cot∝= 1020kgf. $\frac{1}{tan\propto}$ Now we use the value for tan∝, which was already obtained $z = \frac{1020kgf}{1,35}$ where z=756kgf

y = z + F=756kgf+800kgf =1556kgf≈1560kgf

The load on the post foundation is approximately 1560 kgf

To solve a trigonometric equation is important to know some elements associated with these mathematical objects such as the following:

1. Fundamental trigonometric identities

$sen^2x + \cos^2 x = 1 \Rightarrow sen^2x = 1 - \cos^2 x$ y $\cos^2 x = 1 - sen^2x$

a) $sen2x = 2senx.\cos x$
b) $\cos 2x = \cos^2 x - sen^2x$
c) $\tan x = \frac{senx}{\cos x}$, $\cos x \neq 0$
d) $\cot x = \frac{\cos x}{senx}$, $senx \neq 0$
e) $\tan x.\cot x = 1$
f) $sen(x+y) = senx \cdot \cos y + seny \cdot \cos x$
g) $sen(x-y) = senx \cdot \cos y - seny \cdot \cos x$
h) $\cos(x+y) = \cos x \cdot \cos y - sex \cdot seny$
i) $\cos(x-y) = \cos x \cdot \cos y + senx \cdot seny$
j) $\sec(x) = \frac{1}{\cos(x)}$
k) $\csc(x) = \frac{1}{sen(x)}$

Signs of the trigonometric functions in each of the quadrants

	I quadrant	II quadrant	III quadrant	IV quadrant
sen x	+	+	-	-
cos x	+	-	-	+
tan x	+	-	+	-
cot x	+	-	+	-

Reduction formulas in each of the quadrants

If α is the value of the angle of the first quadrant then the value of II quadrant will be given by the expression x=π-α, in the III quadrant it will be given by the expression x=π+α and in the IV quadrant it will be given by the expression
x=2π-α

2.3.1 Procedure for solving trigonometric equations

1- Make all the arguments of the trigonometric functions involved in the equation equal.

2- Obtain only one trigonometric function in the equation (preferably sine or cosine and from these the transformation should be to the one that is linear)

3- Make change of variables.

4- Solve the originated equation.

5- Return to the initial variable.

6- Solve the simple trigonometric equations that are formed.

7- Express the solution.

Example

Solve the following equation in $[0,2\pi]$

$$3(1 - senx) = 1 + \cos 2x$$

$3 - 3senx = 1 + \cos 2x$, as you can see the arguments of the trigonometric functions involved are different, then you have to make them equal. $3 - 3senx = 1 + \cos^2 x - sen^2 x$, Let's transform all the trigonometric functions to sine because it is the one that is linear.

$$3 - 3senx = 1 + 1 - sen^2 x - sen^2 x$$

$$3 - 3senx = 2 - 2sen^2 x$$

Making the change of variables $senx = y$, you have the following equation: $3 - 3y = 2 - 2y^2$ from where $2y^2 - 3y + 1 = 0$

decomposing into factors you have to $(2y - 1)(y - 1) = 0$

$y - 1 = 0$, by obtaining

$$y = \frac{1}{2},\ y = 1$$

Returning to the initial variable is obtained that $senx=\frac{1}{2}$ y $senx=1$

Solving the simple trigonometric equations obtain $senx=\frac{1}{2}$, Let us observe that the sign of $\frac{1}{2}$ is positive then the function $senx$ is positive in the I and II quadrants and therefore we have to give an answer in the I and another in the II quadrant.

In the I quadrant $senx=\frac{1}{2}$ and $x=\frac{\pi}{6}$ applying the formula of reduction of the II quadrant it has to be $x=\pi-\frac{\pi}{6}$, from where it has to $x=\frac{5\pi}{6}$

Solving the other simple trigonometric equation obtained $senx=1$ you have to $x=\frac{\pi}{2}$

Therefore the set solution of the equation is $S=\left\{\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6}\right\}$ when we are in the presence of a trigonometric equation where sine and cosine intervene but both are linear, it is suggested to square both members of the equation to obtain them squared and it is easier to apply identities.

Example

Solve the following equation

$$senx+\sqrt{3}\cos x=2$$

$senx=2-\sqrt{3}\cos x$, by squaring both members, we have to: by squaring both members we have to:

$$(senx)^2=\left(2-\sqrt{3}\cos x\right)^2$$

$sen^2x=4-4\sqrt{3}\cos x+3\cos^2 x$, rom here we can proceed in an analogous way to the previous example, transforming all the functions of the equation to cosine from here we can proceed in an analogous way to the previous example, transforming all the functions of the equation to cosine

$$1-\cos^2 x=4-4\sqrt{3}\cos x+3\cos^2 x$$

$$4\cos^2 x-4\sqrt{3}\cos x+3=0$$

Making the change of variables $\cos x = y$ you have

$$4y^2 - 4\sqrt{3}y + 3 = 0$$

$\left(2y - \sqrt{3}\right)\left(2y - \sqrt{3}\right) = 0$, from where it is obtained

$2y - \sqrt{3} = 0$ which implies that $y = \frac{\sqrt{3}}{2}$, returning to the initial variable,

$\cos x = \frac{\sqrt{3}}{2}$ from where $x = 30^\circ$

But as the function is positive in the first (I) and fourth quadrant (IV) it is necessary to find the other possible solution.

Let's apply then the formula of reduction of the IV quadrant

$x = 360^0 - 30^0$ from where $x = 330^0$

Since squaring is not an equivalent transformation, it is necessary to perform the check because strange solutions could have been introduced. We leave it to the reader to verify that it is only a solution to the equation $x = 30^0$ and all its coterminals. Then, $S = \left\{30^0 + 360^0 k, k \in Z\right\}$

Note: in the solutions strange roots can appear due to the manipulation of the equations when trying to reduce them, for example: it can be a $\cos x = 2$, which we must discard, obviously, because the cosine codomain is limited to $[-1,1]$. Also, we must verify all the answers obtained and accept only those that satisfy the original equation.

Since trigonometric functions repeat their value and sign in two of the quadrants, it should be kept in mind that there will always be at least two different angles in the solution of a trigonometric equation of the form t(x) = b (where t(x): is one of the six trigonometric functions and b: any number that belongs to the image of the function). In addition, because these functions are periodic, it is necessary to add to the solutions obtained a multiple of 360°, that is, k360°, and k is an integer as long as there are no initial conditions for the solution of the equation.

2.3.2 Exercises Proposal

1. Solve the trigonometric equations

$a)\,senx = 0$

$b)\cos x = 0$

$c)\tan x = 0$

$d)\,senx = 1$

$e)\cos x = 1$

$f)\tan x = 1$

$g)\,senx = -1$

$h)\cos x = -1$

$i)\tan x = -1$

$j)\,senx = \frac{1}{2}$

$k)\,senx = -\frac{1}{2}$

$l)\cos x = \frac{1}{2}$

$m)\cos x = -\frac{1}{2}$

2.Solve the trigonometric equations.

$a)\,sen\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$

$b)\,2\tan x - 3\cot x - 1 = 0$

$c)\,3sen^2 x - 5senx + 2 = 0$

$d)\cos^2 x - 3sen^2 x = 0$

$e)\cos 2x = 1 + 4senx$

$f)\,sen\left(2x + 60^0\right) + sen\left(2x + 30^0\right) = 0$

$g)\,sen^2 x - \cos^2 x = \frac{1}{2}$

$h)\cos 8x + \cos 6x = 2\cos 210^0 \cdot \cos x$

$i)\tan 2x = -\tan x$

$j)\,senx + \sqrt{3}\cos x = 2$

$k)\,sen2x = \cos 60^0$

$l)\,4sen\left(x - 30^0\right)\cos\left(x - 30^0\right) = \sqrt{3}$

$m)\,2\cos x = 3\tan x$

$n)\,sen2x \cdot \cos x = 6sen^3 x$

$ñ)\,4sen\frac{x}{2} + 2\cos x = 3$

3. Find the set solution to the following equation:

$$sen4x - 3\cos^2 x + sen^2 x + 1 = 0 \qquad x \in R$$

4. Given the terms. $N = \frac{\tan 135^0 + 5sen450^0}{8 \cdot \cos(-60^0)}$ and $M = 1 + \frac{2 - 4 \cdot \cos^2 x}{sen2x} + \cot x - \tan x$

a) Find the inadmissible values of the term M.

b) Proves that for every value of the domain of M it is fulfilled that

Chapter 3. Exercises for consolidation

1-Solve the following equation:

$$16 \cdot 2^{5senx} = 4^{\cos^2 x}$$

2- You must:

$F(x) = \log_3\left(x - \sqrt{x-1}\right)$ and $G(x) = \dfrac{\cos 2x + senx}{sen^2 x}$

a) a) Solve the equation $F(x) = \tan 45^0$

b) Find all of the intervener's $\left[0; \dfrac{\pi}{2}\right]$ that satisfy $G(x) = 10^{\log 4}$.

3- Find the values of x, with $0^0 \leq x \leq 360^0$, that satisfy the following equation: $10^{\log 16} = 4^{\cos 2x} \cdot 64^{senx}$

4- You have the following expressions:

$H(x) = \dfrac{w \cos x}{\dfrac{senx}{1+\cos x} + \dfrac{1+\cos x}{senx}}$ y $F(x) = senx$

It proves that when $w = 4$, then it is fulfilled $H(x) = F(2x)$ that for permissible values of x.

5-Find the inadmissible values of the variable and prove that trigonometric equality is an identity.

$$\frac{\tan x}{sen2x + 2senx} = \frac{1}{\cos 2x + 2\cos x + 1}$$

6-Find, if they exist, the points of intersection of the following functions, $0 \leq x \leq 2\pi$ to $f(x) = (\tan x)^{\cos 2x}$ and $g(x) = (\cot x)^{senx}$

7- An electric power generator produces a power given by the equation $I = 30 \cdot sen120\pi t$ where t the time in seconds and I is the power amperes. Find the smallest positive time t (with four significant digits) so that $I = -10$ amperes.

8- Determine the actual values of x that satisfy the following equation

$$\log_3\left(\frac{8x^3 + 1}{4x^2 - 2x + 1}\right) + \log_9(4x^2 + 4x + 1) = 4$$

9-Solve the equation.

$$\log_2(3^{x-1}+11)=1+\log_2(3^{x-1}+1)$$

10- Find the values of x ∈ [0,π] that satisfy the equation

$$\log_{(senx+cosx)}(cos2x-sen2x+7cosx+5)=2$$

11- Solve the following equation

$$4^{\log\sqrt{2x^2+7x}-\log\sqrt{x}}=16^{\log\sqrt{x+2}}$$

12-Find the set solution of the following equations

a- $2^{\log x}.2^{\log(2x+7)}-4^{\log(x+2)}=0$, x∈ R
b- $\log(cos2x+cosx)-\log(senx+cosx)=0$, x∈ $[0,\pi]$
c- $9^{\frac{1}{2}+\log_9\sqrt{x+1}}-3^{\log_3\sqrt{x+1}}=5$
d- $\log(1+\sqrt{x+1})-3\log\sqrt[3]{x-4}=0$

13- Determine the solutions of the following equations in the interval [0,2π] a)$sen^3x+cos^3x=cosx$

b)$sen^3x.cosx-senx.cos^3x=\frac{1}{4}$

14- Solve

$$\frac{4^{\log(x+2)}}{2^{\log(x-1)}}=\log[\frac{2}{x}-\frac{4x^2-x+8}{x^2-x}]$$

15- Be the functions defined by the equations

$$f(x)=senx\ y\ g(x)=\frac{x-\pi}{2}$$

For which values of Θ∈ R it is verified that $2^{f(\Theta).cot\Theta}.2^{g(4\Theta+\pi)}=\frac{16}{64^{sen^2\Theta}}$?

16- Be $f(x)=senx\ y\ g(x)=\tan(x)$, calculate all the values of t∈ $[0,2\pi]$, for which it is fulfilled that $f\left(2t-\frac{\pi}{2}\right).\frac{1}{2}f(2t).g(t)=2cos^2t+10[f(t)-1]$

17- Solve

$$(1-tanx)(1+sen2x)=(1+tanx)(1-cos2x),\qquad 0°\leq x\leq 360°$$

18- Find the set solution of the following equations

a) $(1-x)^3=\sqrt{-2x+\sqrt{10(x^2+1)}}.(1-x)^2$

b) $\frac{36^{\log(x-3)}}{(\sqrt{6})^{logx^2}}=6^{\log(x^2+\frac{3}{x})}$

$c)$ $3^{\log_2(x+1)}.9^{\log_2(x-1)}=9^{\log_4(5x+1)}$

$d)$ $\frac{\sqrt{x+3a^2}+\sqrt{x-a^2}}{\sqrt{x+3a^2}-\sqrt{x-a^2}}=\frac{2}{\sqrt{x}}, x>0, x\neq-3a^2, x\neq a^2$

19-Given the function f defined by the equation $f(x) = \sqrt{4x + \sqrt{3x^2 + 52}}$. Determines the actual values of x for which equality is satisfied $2^{f(x)} = 4.2^x$

20- Solves

$\sqrt{x^2 + 7x + 7} + \sqrt{x^2 + 7x - 9} = 8, x \in R$

21-Find the set solution of the following equations

$a)\ \dfrac{1}{x - \sqrt{4 - x^2}} - \dfrac{1}{x + \sqrt{4 - x^2}} = 1$

$b)\ \frac{a+\sqrt{a^2+x^2}}{x} + \frac{x}{a+\sqrt{a^2+x^2}} = 2\sqrt{2}$

$c)\ (a^2 - b^2)x^2 - 4abx = a^2 - b^2,\ |a| \neq b$

$d)\ \dfrac{a^2}{4(x^2 - a^2)} - \dfrac{2x}{x + a} + \dfrac{x}{a - x} = 0,\ |x| \neq a, a \neq 0$

22- Find the solutions to the following equations

$a)\ sen\left(2x + \frac{\pi}{4}\right) = sen(3x + \frac{\pi}{3})$

$b)\ senx.tanx + 1 = senx + tanx$

23- Solve

$a)\ \log_{\frac{1}{5}} \log_5 \sqrt{5x} = 0$

$b)\ \dfrac{log2 + \log(4 - 5x - 6x^2)}{log\sqrt[3]{2x - 1}} = 3$

$c)\ x^2 \log_6(5x^2 - 2x - 3) - x \log_{\frac{1}{6}}(5x^2 - 2x - 3) = x^2 + x$

$d)\ 5^{1+2x} + 6^{1+x} = 30 + 150^x$

$e)\ \sqrt{2x\sqrt[3]{4^x.0{,}125^{\frac{1}{x}}}} = 4\sqrt[3]{2}$

$f)\ (2 + \sqrt{3})^x + (2 - \sqrt{3})^x = 4$

$g)\ \dfrac{1+sen2x}{sex+cosx}=\sqrt{2}\cos(\frac{\pi}{4}-x)$

24- Be the equation $x^2+2px+q$ whose roots are the real numbers $x_1\, y\, x_2$. Build the equation whose roots are $\frac{x_1}{x_2}$ y $\frac{x_2}{x_1}$

25- Be the equation $2x^2+(k+1)x+3(k-5)=0$. Find all the values that k must take for it to be fulfilled:

a) All roots are real and equal

26- Hallar k en la ecuación $x^2+kx-9=0$ de modo que las raíces x_1y x_2 de la ecuación sean reales e iguales.

27-Solve the following equation

a) $2x^2-6x+\sqrt{x^2-3x+6}=24$

28-Investigate for which values of λ∈ R the roots x_1y x_2 of the equation

$(\lambda-1)x^2-2(\lambda-2)x+\lambda-4=0, \lambda\neq 1$

a) They are the same

29- Be the terms A(x)=$\sqrt{cos2x}$ y B(x)=$\sqrt{4-\frac{1}{2}sen2x.tanx}$

a) Calculate B($\frac{\pi}{6}$)

b) For which values of x∈ $(-3\pi; 6\pi]$ Is it verified that the term B(x) exceeds the term A(x) by 1?

30-There is a composite body formed by a straight circular cylinder of radius r and height h and a hemisphere of equal radius

a) If h= 3,00m. Calculate r so that the body volume is 6πm^3.

b) If the height of the cylinder is 2.50m. What should be the length of the radius of the cylinder so that the lateral area of the body is equal to 2πm^2

31- Be the functions defined by the equations

$$f(x) = (senx - 2)^2 + sen(\frac{\pi}{2} - 2x)$$

$$g(x) = \frac{3cos^2 2x - cos4x - sen4x.cot2x}{1 - cos^2(\frac{\pi}{2} - 2x)}$$

$$h(x) = \frac{sen(\pi - 2x)}{\cos(2x + 4\pi)}$$

a) Calculate $h\left(\frac{\pi}{6}\right) + tan780°$

b) Determine the coordinates of the points where the $f(x)$ intersects the abscissa axis $(0 \leq x \leq 2\pi)$

32-Determine the values of x ∈ [0;2π) that comply with the equality

$$\log_3\left(3^{\sqrt{4cos2x - cosx}} + 18\right) - 3 = 0$$

33-Be the function $h(x) = sen2x.tanx - kcosx - 1$, si $h\left(\frac{\pi}{3}\right) = -2$, calculates the zeros of the given function in the interval [-$\frac{\pi}{2}$; 2π)

34-A chemist makes two mixtures of sodium hypochlorite and water with concentrations of 20% and 40% respectively. Determine the amount that needs to be used from each of the mixtures to obtain 30 grams with a concentration of 32%.

35-The telegraph poles along a railroad track are evenly spaced. If each space were increased by 51m there would be 15 fewer posts per kilometer. How many posts are there per kilometer and what is the spacing?

36-In a certain country the income of an individual is 5000 pesos for one year. After deducting the state tax, which is 1% less than the state rate, his net income is 4656 pesos. Determine the percentage you pay to the state.

37-Determine the set solution of the following equation

$$\sqrt{\frac{x-2}{x+3}}+\sqrt{\frac{x+3}{x-2}}=2\frac{1}{6}$$

38- For which value of m the equation

$2mx^2-(1+m)x+m-1=0$, you have that the quotient of your roots is 1?

39- Let be the function defined by the equation $f(x)=3x^2-(k+1)x+2, k\in R$

a) Determine the value of k so that the difference between the zeros of the given function is 2

40-Solve the following equation

$$\log_5(5^x+125)=2\log_{25}6+1+\frac{1}{2x}$$

41-Se the quadratic equations of variable x and whose definition domain is the set of real numbers

$(7a-2)x^2-(5a-3)x+1=0\ y\ 8bx^2-(4b+2)x+2$. Determine the value of the parameters $a\ y\ b$ so that the given equations are equivalent.

42- 9L were taken from a barrel that was full of wine, replacing them with water, then 9L were taken from the mixture, which were replaced with water. The amount of wine remaining in the barrel and the amount of water are in the reason $\frac{19}{9}$. Calculate barrel capacity.

43-Be $f(x)=3^x$. Find all the values of t for which it is fulfilled that

$$\sqrt{3+6(1-f(t))}=\sqrt{f(t+1)}$$

44-Find the set solution of the following equation

$$\log\sqrt{4^x+6}+\frac{1}{\log\sqrt{4^x+6}}=2\frac{1}{2}$$

45-Be $f(x)=\log_2(9^{x+1}+7)y\ g(x)=2+2\log_4(3^{x-1}+1)$. For which values of $x\in R$ the reason between $f\ y\ g$ is equal to unity?

46- The geometric mean and the arithmetic mean of two numbers that differ in 32 are in the reason $\frac{3}{5}$. Find the numbers.

47-Find the set solution of the following equations

$$a)\ \frac{(senx + cosx)^2 - (cos2x + 3)}{1 - tanx} = sen2x$$

$$b)\ (\sqrt{5})^{4sen^3x - sen2x.tanx} - 25^{\frac{5}{2}\cos(\pi - x)+1} = 0$$

$$c)\ a^{-1} + b^{-1} + x^{-1} = (a + b + x)^{-1}, a, b \neq 0, a + b \neq -x$$

48-Be the functions

$$f(x) = sen2x.cotx\ y\ g(x) = -\cos\left(\frac{\pi}{2} - x\right) + 1$$

a) For which values of x$\in R$ it is verified that $(\sqrt{2})^{f(x)} - 2^{0,25+g(x)} = 0$?

49-For which values of the equation $x^2 - 2(a-5)x + (a+1)(a-1) = 0$

a) It has a unique real solution

50-See the trigonometric terms

$A(x) = 4^{tanx.cotx}.(\frac{1}{16})^{senx}$ y $B(x) = 4^{sen2x}$

a) Find the actual values of $x \in [-5\pi; 4\pi)$ so that it is fulfilled that $\frac{A(x)}{B(x)} = 1$

52-Solve the following equations

$$a)\ (x^2 - x + 1)^4 - 10x^2(x^2 - x + 1)^2 + 9x^4 = 0$$

$$b)\ \frac{\sqrt{x^2-5x+6}}{2} = \frac{12-\sqrt{x^2-5x+6}}{\sqrt{x^2-5x+6}}$$

$$c)\ 2\sqrt{x} + \sqrt{x^2 - 9} + \sqrt{x^2 - 1} + \sqrt{3x - 5} = 0$$

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